Mathellaneous by Daniel Mathews

A beautiful sequence

For one or another reason the following sequence stumbled upon me one day; well, I wouldn't do it the dishonour of saying I stumbled on *it*.

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, \ldots.$$

We'll call the sequence $\{a_n\}$. It is an increasing sequence of positive integers, with increment of 1 or 2 each step, arranged somewhat sporadically. If we look a little more closely at these increments, we see that they are not random at all.

1		3		4		6		8		9		11		12		14		16
	2		1		2		2		1		2		1		2		2	

The 2's occur in positions numbered $1, 3, 4, 6, 8, 9, \ldots$. So $a_{n+1} = a_n + 2$ if and only if n occurs in the sequence; otherwise $a_{n+1} = a_n + 1$. This is sufficient to define our sequence, starting from $a_1 = 1$: a remarkable property of self-reference.

We can now take the complement of this sequence, i.e. the set of all positive integers *not* in $\{a_n\}$, arrange them in increasing order, and write them out. Let this sequence be $\{b_n\}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25
b_n	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41

So we have the nice relationship $a_n + n = b_n$. In fact this gives another way to define the sequences, recursively: set $a_1 = 1$, $b_1 = 2$. Then let a_n be the least integer not used so far, and let $b_n = a_n + n$.

Now examination of the pairs (a_n, b_n) shows some remarkable properties to those familiar with the Fibonacci and related sequences. In fact, $(1, 2)$, $(3, 5)$, $(8, 13)$, $(21, 34)$, $\ldots$ are pairs of consecutive Fibonacci numbers (recall $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$). If we continue onward, we obtain all Fibonacci numbers in such pairs. Where do such Fibonacci pairs occur? In positions $1, 2, 5, 13, \ldots$ — which is every second Fibonacci number $(1, (1), 2, (3), 5, (8), 13, \ldots)$.

Fibonacci-type properties do not end there. Start from another pair, say $(4, 7)$. If we apply the Fibonacci recurrence to 4 and 7 as starting values, we obtain $4, 7, 11, 18, 29, 47$, $\ldots$ — all occurring as pairs (a_n, b_n). Indeed, for any pair (a_k, b_k), the Fibonacci-type sequence starting with a_k, b_k occurs completely in pairs (a_n, b_n). And the positions where these pairs occur are again the numbers which are every second term of the Fibonacci-type sequence.

Thus our sequences (a_n, b_n) actually *partition* the positive integers into a set of disjoint Fibonacci-type sequences.

1,	2,	3,	5,	8,	13,	21,	34,	...
4,	7,	11,	18,	29,	47,	76,	123,	...
6,	10,	16,	26,	42,	68,	110,	178,	...
9,	15,	24,	39,	63,	102,	165,	267,	...
12,	20,	32,	52,	...				
...								

It's not obvious that such a partition exists, at first thought, nor is it easy to construct these sequences from scratch. (Observe that a greedy algorithm, taking least unused numbers at each stage, doesn't work.) Is this the only such partition?

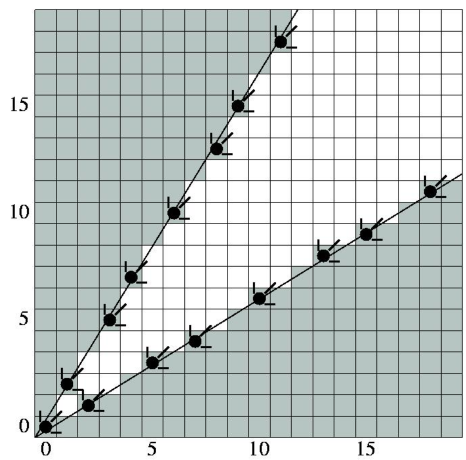
Consider, now, where these Fibonacci-type sequences start. We have $(a_1, b_1) = (1, 2)$ starts the first sequence, $(a_3, b_3) = (4, 7)$ starts the second, (a_4, b_4) starts the third, then (a_6, b_6), then (a_8, b_8). In general, (a_n, b_n) starts one of these Fibonacci-type sequences if and only if n occurs in our sequence $\{a_n\}$, in another startling display of self-reference.

This gives us another construction for a_n, b_n. Start with $(a_1, b_1) = (1, 2)$. We extend this to a full Fibonacci sequence and place it at terms numbered by every second Fibonacci number, so $(a_2, b_2) = (3, 5)$, $(a_5, b_5) = (8, 13), \ldots$. Having done this, we find the least unnumbered spot and set $a_3 = 4$, the least positive integer unused so far. Then we set $b_3 = a_3 + 3 = 7$, and proceed to fill in all the terms of this Fibonacci-type sequence by a similar rule. Continue inductively to define a_n, b_n.

The considerations above give the equation $a_n + b_n = a_{b_n}$. In fact there are more relationships of this form ([3], [10], [13], [17]):

$$a_{a_n} = b_n - 1, \qquad a_{b_n} = a_n + b_n, \qquad b_{a_n} = a_n + b_n - 1, \qquad b_{b_n} = a_n + 2b_n.$$

Our sequence arises not only from idle numerology, but in various concrete situations. Consider the lattice points in the positive quadrant of the plane, $(m, n) \in \mathbb{Z}^2$, $m, n \geq 0$. I have mathematical lighthouses which shine trifurcated light — upwards, rightwards and diagonally right-up. (i.e. along vectors $(1, 0), (0, 1), (1, 1)$). I want to light up the entire quarter-infinite lattice. (A lighthouse lights its own point, and the light shines through other lighthouses.) Call a point "dark" or "lit" accordingly.



One way to proceed is as follows. Obviously, there must be a lighthouse at $(0, 0)$. This shines on every point $(n, 0), (0, n)$ and (n, n). Then, the two dark points closest to the origin are $(1, 2)$ and $(2, 1)$. Place lighthouses at these points, and continue in this fashion. The next points to receive lighthouses are then $(3, 5), (5, 3)$, then

$(4, 7), (7, 4)$, $(6, 10), (10, 6)$, ... A little reflection will show that the nth pair of points $(x_n, y_n), (y_n, x_n)$ satisfy $y_n = x_n + n$, and x_n is the least x-coordinate thus far unused. So the points with lighthouses are exactly the points (a_n, b_n) and (b_n, a_n).

The lighthouses appear to lie along two "lines" enclosing a "cone". Points inside the cone receive light travelling diagonally from a lighthouse; those outside receive light from just one lighthouse, travelling horizontally or vertically. For this reason we will call the cone a "light cone", relativity theory notwithstanding. This is in a sense the most "energy efficient" way to place our lighthouses: it is the only way to light up the quarter-plane such that no two lighthouses lie on the same row, column or diagonal.

This situation is the set-up for a simple 2-player game, invented by Rufus P. Isaacs, first described in [2] and named "Corner the Lady" by Martin Gardner [8]. It can be played online at [23]. We have a quarter-infinite chessboard, on which is placed one queen. Two players take it in turns to move the queen. However, unlike a normal queen, this piece only moves towards the origin (i.e. down, left, or diagonally down-left), any number of squares. Players cannot pass; the player who moves the queen to the origin wins.

It's not difficult to see that this chessboard game is closely related to a variant of the game of Nim, known as *Wythoff's Nim* [20] and played in China as *tsianshidsi* ("choosing stones"). It can also be played online at [24]. We have two non-negative integers, A and B. Two players take it in turns to decrease numbers. On their turn, a player may decrease A by any amount provided it remains a non-negative integer. Or they may decrease B in similar fashion. Or, they may decrease *both* A and B by the *same* amount, again provided A, B remain non-negative integers.

Who has a winning strategy? The answer depends on the initial state, i.e. the initial position of the queen or values of A and B. The game satisfies criteria known to game theorists showing that the game is completely soluble. It's quite clear, for a start, that if you move the queen to the bottom row or leftmost column, you are going to lose — your opponent will move the queen to the origin next turn. Similarly, if you move onto the main diagonal $y = x$. In fact, you can see that the closest "safe" points to the origin are $(1, 2)$ and $(2, 1)$. If you move the queen to one of these points, then your opponent must move to $(1, 1)$, $(1, 0)$, $(0, 1)$, $(2, 0)$ or $(0, 2)$. From any of these squares you can move to $(0, 0)$ and win the game.

Extrapolating and following similar reasoning, one can show that the only way to avoid loss, against a sufficiently intelligent opponent, is to move to a square with a lighthouse, i.e. a square of the form (a_n, b_n) or (b_n, a_n). From a safe point with a lighthouse, you can only move to unsafe points. And from every unsafe point you can reach the shelter of a lighthouse. So if the queen starts on a lighthouse, player 2 has a winning strategy. Otherwise player 1 has a winning strategy.

In fact this diagram has great relevance to another game, this time more purely number-theoretic. The *Game of Euclid* ([4], [7]) is a game for two players, and we start with two positive integers. On their turn, a player may subtract any (positive) multiple of the smaller number from the larger, provided that both numbers remain non-negative. So, if the game starts with $(12, 5)$, then the first player could move to $(7, 5)$ or $(2, 5)$. The player who reduces one of the numbers to zero wins. The object of the game, then, is to perform a complete Euclidean algorithm on the two initial numbers (hence the name!).

We can analyse the game on our lighthouse chessboard again. Moving to $(n, 1)$ or $(1, n)$ is a bad idea, as your opponent will win on the next move. The closest "safe" points to the origin are $(2,3)$ and $(3,2)$, from which your opponent must move to $(2,1)$ or $(1,2)$, from which you have an assured victory.

Indeed, one can prove that the light cone, minus the main diagonal, is precisely the "safe" region for this game: to win against a sufficiently intelligent opponent, you must always move to this region. From outside the light cone at a point say (x,y) with $x < y$, you can always move into the light cone minus diagonal (if x is not a factor of y), or to $(x,0)$ (otherwise) and win. This follows from the fact that the height of the light cone in column x is precisely x squares. (Count points with lighthouses as outside the light cone.) And from the safe region, you can only move into unsafe lands. This gives a geometric interpretation which seems more intuitive than the purely algebraic approach of [4].

We now turn to consider the two "lines" of lighthouses bounding the light cone. They do indeed lie on a "rounded-down" line.

Recall that the Fibonacci numbers satisfy

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887498948482045868343 65\cdots,$$

the golden ratio. In fact every Fibonacci-type sequence of positive integers has this property: this follows from the fact that they obey the same recurrence, and that the two roots of the characteristic equation are ϕ and $1-\phi$ (which is negative). Therefore

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \phi,$$

and we expect the lighthouses to lie asymptotically on the lines $y = \phi x$, $y = x/\phi$.

We can do even better than this. If $\lfloor x \rfloor$ denotes the integer part function, it is actually true that $a_n = \lfloor n\phi \rfloor$ and $b_n = \lfloor n\phi^2 \rfloor$. So the lighthouses lie on the lines $y = \phi x$, $y = x/\phi$ to the nearest integer.

The sequences a_n, b_n give an example of *Beatty sequences*, so-named after Beatty's beautiful theorem, first proposed as a problem in the American Mathematical Monthly in 1926 [1]: if α, β are two positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the two sequences

$$\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \lfloor 4\alpha \rfloor, \ldots \quad \text{and} \quad \lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \lfloor 4\beta \rfloor, \ldots$$

partition the integers. That is, they are disjoint but include every positive integer.

The proof of Beatty's theorem is truly elegant, considering perhaps the unexpectedness of the result. This proof is originally due to Ostrowski and Hyslop and can be found in [6], [12], [16]. There is also a more direct proof by contradiction, in [7].

Consider how many members there are of each sequence less than some positive integer N. Clearly there are $\lfloor N/\alpha \rfloor$ in the first sequence, and $\lfloor N/\beta \rfloor$ in the second. So in total there are

$$\left\lfloor \frac{N}{\alpha} \right\rfloor + \left\lfloor \frac{N}{\beta} \right\rfloor \tag{1}$$

numbers in the two sequences less than N. Now we know $\frac{N}{\alpha} + \frac{N}{\beta} = N$, but rounding down takes off more than 0 (since α, β are irrational) and less than 2 (since we round down twice). Thus the sum (1) is $N-1$, and there are $N-1$ numbers in the two sequences less than N.

Thus there is 1 number present less than 2; it must be 1. Then there are 2 numbers present less than 3; so they must be 1 and 2. Proceeding in this fashion, it follows that every positive integer occurs exactly once in our two sequences.

Being so surprising and elementary, many mathematicians have attempted generalisations of Beatty's theorem. Uspensky [19] proved that you cannot have *three* numbers α, β, γ such that the sequences $\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor, \lfloor n\gamma \rfloor$ together contain each integer exactly once, under *any* conditions. Nor can you have 4 or more such numbers. R.L. Graham gives a simple proof in [9].

While on this theme, consider the sequences $f_n = a_n - n$ and $g_n = b_n - n$. (Actually g_n and a_n are the same.)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f_n = a_n - n$	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9
$g_n = b_n - n$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	23	25

We see that f_n is equal to the number of terms of the sequence $\{g_k\}$ such that $g_k < n$. And reciprocally, g_n is equal to the number of terms of the sequence $\{f_k\}$ such that $f_k < n$.

In fact the same applies, starting not just with a_n, b_n, but starting with *any* two increasing sequences partitioning the positive integers. The reader is encouraged to take any two such sequences $\{p_n\}, \{q_n\}$, form the sequences $\{p_n - n\}, \{q_n - n\}$, and see what happens! This much, and more, is a remarkable theorem of Lambek and Moser [15], treated also in [12].

Finally, consider the Fibonacci base number system, also known as Zeckendorf arithmetic ([18], [22]). This is like other number bases, with digits 0 and 1, except that the place values are $1, 2, 3, 5, 8, 13, \ldots$ and we add the restriction that *no two 1s may be adjacent*. We can denote the Fibonacci base by a subscript ϕ; indeed it functions similarly to a "base-ϕ" system. The first few numbers written in Fibonacci base are

$$\begin{aligned}
0_{10} &= 0_\phi \\
1_{10} &= 1_\phi \\
2_{10} &= 10_\phi \\
3_{10} &= 100_\phi \\
4_{10} &= 101_\phi \\
5_{10} &= 1000_\phi \\
6_{10} &= 1001_\phi \\
7_{10} &= 1010_\phi \\
8_{10} &= 10000_\phi \\
9_{10} &= 10001_\phi \\
10_{10} &= 10010_\phi
\end{aligned}$$

The numbers ending in 0 in the Fibonacci base are $0, 2, 3, 5, 7, 8, 10, \ldots$. These numbers are precisely $a_n - 1$. It is an interesting exercise to devise a strategy for winning Wythoff's Nim purely in terms of these Fibonacci representations ([8], [17]). Investigations of the Fibonacci base and this game have run quite deep (e.g. [3]).

We conclude our tour of some of the properties of this remarkable sequence at this point. The diligent reader should be able to prove all the assertions made here. Surely there is much more of interest to reward the keen investigator.

This sequence $\{a_n\}$, its complementary sequence $\{b_n\}$, and associated mathematical objects cannot help but remind us of James Jeans' famous assertion that God is a mathematician. The appearance of such elementary objects in so many different realms, with such unexpected and elegant results, conjures up the sense of awe and curiosity that is at the heart of mathematical inquiry.

I will leave the reader with a problem I met through mathematical olympiad encounters. It is from the Iranian olympiad training programme of 2000; thanks to Angelo Di Pasquale for locating the source. The answer will come as no surprise.

Problem. Suppose $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a function such that $f(1) = 1$ and

$$f(n+1) = \begin{cases} f(n) + 2 & \text{if } n = f(f(n) - n + 1) \\ f(n) + 1 & \text{otherwise} \end{cases}$$

Prove that $f(f(n) - n + 1) \in \{n, n + 1\}$ and find the function f.

References

[1] S. Beatty, *Problem 3173*, Amer. Math. Monthly **33** (1926), 159.
[2] C. Berge, *The theory of graphs and its applications*, (Wiley 1961), chapter 6.
[3] M. Bicknell-Johnson, *Generalized Wythoff numbers from simultaneous Fibonacci representations*, Fibonacci Quart. **23** (1985), 308–318.
[4] A.J. Cole and A.J.T. Davie, *A game based on the Euclidean algorithm and a winning strategy for it*, Math. Gaz. **53** (1969), 354–7.
[5] I.G. Connnell, *A generalization of Wythoff's game*, Canad. Math. Bull. **2** (1959), 181–190.
[6] H.S.M. Coxeter, *The golden section, phyllotaxis, and Wythoff's game*, Scripta Math. **19** (1953), 135–143.
[7] A. Engel, *Problem-Solving Strategies*, (Springer New York 1998), chapters 6 & 13.
[8] M. Gardner, *Wythoff's nim* in *Penrose tiles to trapdoor ciphers*, (Freeman New York 1989), chapter 8.
[9] R.L. Graham, *On a theorem of Uspensky*, Amer. Math. Monthly **70** (1963), 407–409.
[10] V.E. Hoggatt Jr, M. Bicknell-Johnson and R. Sarsfield, *A generalization of Wythoff's game*, Fibonacci Quart. **17** (1979), 198–211.
[11] J.C. Holladay, *Some generalizations of Wythoff's game and other related games*, Math. Mag. **41** (1968), 7–13.
[12] R. Honsberger, *Ingenuity in Mathematics*, (Random House/Singer School Division 1970; MAA 1998), essay 12.
[13] A.F. Horadam, *Wythoff pairs*, Fibonacci Quart. **16** (1978), 147–151.
[14] H.E. Huntley, *The divine proportion*, (Dover 1970).
[15] J. Lambek and L. Moser, *Inverse and complementary sequences of natural numbers*, Amer. Math. Monthly **61** (1954), 454–458.
[16] A. Ostrowski and J. Hyslop, *Solution to Problem 3177*, Amer. Math. Monthly **34** (1927), 159.
[17] R. Silber, *A Fibonacci property of Wythoff pairs*, Fibonacci Quart. **15** (1977), 85–88.
[18] G.J. Tee, *Russian peasant multiplication and Egyptian division in Zeckendorf arithmetic*, Aust. Math. Soc. Gaz. **30** (2003), 267–276.
[19] J.V. Uspensky, *On a problem arising out of the theory of a certain game*, Amer. Math. Monthly **34** (1927), 516–521.
[20] I.I. Wythoff, Nieuw Arch. v. Wisk. **7** (1907) 199.
[21] A.M. Yaglom and I.M. Yaglom, *Challenging mathematical problems with elementary solutions*, Vol. **2** (1967), 20, 105.
[22] E. Zeckendorf, *Represéntation des nombres naturels par une somme de nombres de Fibonacci ou les nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **41** (1972), 179–82.
[23] http://www.cut-the-knot.org/pythagoras/withoff.shtml [sic]
[24] http://sciris.shu.edu/~borowski/Wythoff

Department of Mathematics and Statistics, The University of Melbourne, VIC 3010
E-mail: D.Mathews@ms.unimelb.edu.au