# Notes on Eliashberg's 1989 paper, "Classification of overtwisted contact structures on 3-manifolds"

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## 1 Introduction

Eliashberg in 1989 in [1] triumphed over overtwisted structures. They are now completely classified.

Let $M$ be an oriented connected 3-manifold. Take a basepoint $p \in M$ and an embedded disc $\Delta \subset M$ centred at $p$. Let $\mathrm{Distr}(M)$ denote the space of all tangent 2-plane distributions on $M$, with one extra condition: we fix the distribution at the point $p$. We give $\mathrm{Distr}(M)$ the $C^\infty$ topology. We let:

- $\mathrm{Cont}(M)$ be the subspace of $\mathrm{Distr}(M)$ which consists of (positive) contact structures;
- $\mathrm{Cont}^{OT}(M)$ be the subspace of $\mathrm{Cont}(M)$ consisting of all overtwisted structures which have the disc $\Delta \subset M$ as the standard overtwisted disc.

There are then obvious inclusions

$$\begin{aligned} i &: \mathrm{Cont}(M) \longrightarrow \mathrm{Distr}(M) \\ j &: \mathrm{Cont}^{OT}(M) \longrightarrow \mathrm{Distr}(M). \end{aligned}$$

**Theorem 1.1** *The inclusion $j : Cont(M)^{OT} \longrightarrow Distr(M)$ is a homotopy equivalence.*

For $M$ an open manifold this follows from an old (1969) theorem of Gromov in [2].

This theorem subsumed the theorem of Lutz [3] that $i_* : \pi_0(\mathrm{Cont}(M)) \longrightarrow \pi_0(\mathrm{Distr}(M))$ is surjective. Very much subsumed it!

This basically means that the classification of overtwisted contact structures is the same as the classification of 2-plane fields on $M$. If you take a homotopy class of 2-plane fields, then there's an overtwisted contact structure among them. And, if you have two contact structures in the same homotopy class of 2-plane fields, then they are homotopic through contact plane fields, i.e. they are isotopic contact structures. It follows from Gray's theorem that they're isomorphic.

## 2 Sketch of the proof

The proof has some nice ideas, but also some hard analysis. (Hard as in many $\epsilon$'s and many things to keep control of, but still elementary in nature.)

Theorem 1.1 is really about an extension problem. We need to find a homotopy inverse; from each 2-plane distribution we need to find a contact structure. Their compositions must be homotopic to the identity. In one direction this will be easy, because homotopies of contact structures are obviously homotopies of 2-plane distributions. But in the other direction we have something to prove: given a homotopy of 2-plane distributions, we need to find a homotopy of contact structures. This can be considered an extension problem: in fact Eliashberg proves the following theorem, which "immediately" implies theorem 1.1.

**Theorem 2.1** *Let:*

*(i) $M$ be a compact 3-manifold;*

*(ii) $A \subset M$ be a closed subset such that $M \backslash A$ is connected (we are thinking of $A$ as a 1-skeleton of a simplicial decomposition of $M$);*

*(iii) $K$ be a compact space (parameter space for the homotopy)*

*(iv) $L \subset K$ a closed subspace (smaller parameter space; we want to extent from $L$ to $K$).*

*(v) $\xi_t$ be a family of 2-plane distributions on $M$ defined for all $t \in K$. For $t$ in the smaller parameter space $L$, $\xi_t$ is contact on $M$. And for all $t$ in the total parameter space $K$, $\xi_t$ is contact on the closed subset $A$.*

*(So we have a homotopy extension problem: the homotopy is defined on $A$, and partially defined on $M$; we need to extent it.) Suppose that there is an disc $\Delta$ in $M$ which is always a standard overtwisted disc, for all $t \in K$: rigorously,*

*suppose there exists an embedded 2-disc* $\Delta \subset M \backslash A$ *such that for all* $t \in K$, $\xi_t$ *is contact near* $\Delta$ *and* $(\Delta, \xi_t)$ *is equivalent to the standard overtwisted disc.*

*Then the homotopy problem can be solved! That is, there exists a family* $\xi'_t$ *of contact structures on* $M$ *for all* $t \in K$ *such that*

*(i) for all* $t$ *in the total parameter space* $K$, $\xi'_t$ *coincides with* $\xi_t$ *near* $A$*;*

*(ii) for all* $t$ *in the smaller parameter space* $L$, $\xi'_t$ *coincides with* $\xi_t$ *everywhere on* $M$*.*

*(iii) the family* $\xi'_t$*, over all* $t$ *in the total parameter space* $K$*, can be connected with* $\xi_t$ *through a homotopy, which is fixed on* $A \times K \cup M \times L$*.*

Because it's a homotopy extension problem, it is sufficient to consider $M$ a compact subset of $\mathbb{R}^3$. $M$ can be covered by such sets; and we just extend repeatedly over them. Then for any two-plane field $\xi$, we have a Gauss map $M \longrightarrow S^2$ and we define the *norm* $||\xi||$ of a 2-plane distribution to be the maximum of the derivative of the Gauss map. The norm of the distribution is the fastest speed at which the plane turns.

# 3 Step I: Construct near a 2-skeleton

The first step is to construct the contact structure (and homotopy!) near the 2-skeleton of a general simplicial complex for $M$. We can effectively take $M, A$ to be simplicial. We will take a very fine subdivision so that the diameter goes to zero while all relevant angles are bounded below, and while the minimal distance between disjoint simplices (relative to the diameter) is bounded below. To be precise! $P$ is a simplicial complex.

(i) $\alpha(P)$ is the minimal angle between non-incident 1- or 2-simplices which have a mutual vertex;

(ii) $d(P)$ is the maximal diameter of a simplex of $P$;

(iii) $\delta(P)$ is the minimal distance between two 0-, 1- or 2-simplices without mutual vertices.

A lot of subdivision and a little perturbing and thought gives us

**Lemma 3.1** *There exists a sequence of general subdivisions* $P_i$ *of* $P$ *such that* $d(P_i) \longrightarrow 0$ *while* $\delta(P_i)/d(P_0)$ *and* $\alpha(P_i)$ *are bounded below.*

In effect, we can make $d(P)$ arbitrarily small without worrying about the other parameters.

Now, we will find a simplicial complex $P$ such that we can do the extension in a neighbourhood of its 2-skeleton. To be precise:

**Lemma 3.2** *Let* $M, A, K, L, \xi_t$ *be as above. (Although now* $M$ *is a compact subset of* $\mathbb{R}^3$*.) Then there exists a general simplicial complex* $P$ *containing* $M$ *and a family of distributions* $\xi'_t$ *(on* $M$*) defined for all* $t$ *in the total parameter space* $K$*, such that:*

*(i) for all $t$ in the parameter space $K$, $\xi'_t$ is $C^0$-close to $\xi_t$;*

*(ii) $\xi'_t$ agrees with $\xi_t$ on $A \times K \cup M \times L$;*

*(iii) $\xi'_t$ is contact on a neighbourhood on the 2-skeleton of $P$. In fact it is contact in a "uniform" way: there exists an $\epsilon > 0$ such that $\xi'_t$ is contact in an $\epsilon \cdot d(P)$-neighbourhood of the 2-skeleton. Here $\epsilon$ depends only on $\alpha$ and $\delta/d$.*

*(iv) The norm $||\xi'_t||$ is bounded as $||\xi'_t|| \leq C||\xi_t|| + D$, for some universal constants $C$ and $D$.*

Note that $\xi'_t$ is actually defined on $M$, but we really only require something of it on $A$ (once this is done, all we require is to extend it to $M$ while remaining $C^0$-close to $\xi_t$; this is easy).

We build up the extension from $A$ to $M$, simplex by simplex. What makes the extension difficult is if $\xi_t(x)$ varies too much, over $t$, or over $x$. The variation over $t \in K$ can be dealt with directly: taking a subdivision of $K$ now, we can assume that for any $t, t' \in K$ the planes $\xi_t$ and $\xi_{t'}$ are always close. (We will have to make sure that we can keep extending over all parts of $K$!) And variation over $x \in M$ is dealt with by a division into cases. A 1-simplex $\sigma$ is called *special* if at some point $x$ of $\sigma$ and for some $t \in K$, $\xi_t$ is too close to being perpendicular to $\sigma$. A 2-simplex $\sigma$ is called *special* if at some point $x$ of $\sigma$ and for some $t \in K$, $\xi_t$ is too close to being parallel to $\sigma$.

By the above lemma 3.1, we can make the diameter $d$ arbitrarily close to zero while keeping $\alpha$ bounded below, which makes special simplices isolated. ($\xi$ doesn't change much in $t$ since we subdivided $K$; $\xi$ doesn't change much on a simplex $x$ since $d$ is small; other nearby simplices differ a definite angle since $\alpha$ is bounded below; so a nearby simplex will not be special.)

The trick is to consider a 2-dimensional foliation $\mathcal{F}_\sigma$ near each simplex $\sigma$ over which we wish to extend. $\mathcal{F}_\sigma$ is a foliation of a neighbourhood of $\sigma$ (not depending on any $t$, but of course varying with $x$ by planes which are perpendicular to for $\xi_t(x)$, for one (random) value of $t$. If $\sigma$ is a 1-simplex, we require $\mathcal{F}_\sigma$ to be parallel to $\sigma$; if $\sigma$ is a 2-simplex, we require $\mathcal{F}_\sigma$ to be perpendicular to $\sigma$. (This is sufficient to define a plane at each point; and clearly it's integrable. There is ambiguity when $\sigma$ parallel/perpendicular to $\xi_t$; but this only occurs at special points in special simplices!) On each (2-dimensional) leaf of $\mathcal{F}_\sigma$, for each $t \in K$ we may obtain a 1-dimensional foliation by intersecting with the 2-plane field $\xi_t$. (By construction $\xi_t$ is always transverse to $\mathcal{F}_\sigma$.) We perturb $\xi_t$ along these 1-dimensional leaves. It turns out that this can be done provided that for each 1-dimensional leaf $l$ we have $\pi_1(l, A \cap l) = 0$. Why is this? The idea is that along a tangent curve $l$, a plane field has a standard form and if you can make it "always twist in the same direction", it's contact. The $\pi_1$ condition allows you to put a twist in continuously. (We will not yet worry that $\xi'_t$ is $C^0$-close!) The $\pi_1$ condition is satisfied for special simplices because special simplices are isolated, and have at most one face belonging to $A$.

Next we turn to neighbourhoods of non-special 1- and 2-simplices. For sufficiently small neighbourhoods, the $\pi_1$ condition will be satisfied; there are less complications since the angle between $\xi_t$ and $\sigma$ is never too close (for any $t$ or $x$). Again we can perturb $\xi_t$ to $\xi_t'$ using the foliation $\mathcal{F}_\sigma$.

Provided we take $d$ sufficiently small, we can get $\xi_t'$ sufficiently $C^0$-close to $\xi_t$. The norm $||\xi_t'||$ may increase, but only linearly. So we can continue extending over all the simplices until we are done. And then we can continue over different subdivisions of $K$. By the end our neighbourhood will be very small indeed; but it is still a subdivision. However, if we keep track of the geometry, it only depends on the geometry of the simplices, namely it is of the form required.

# 4 Step II: A contactization with holes

Now we've done our extension on a neighbourhood of a two-skeleton. This really amounts to the whole manifold, minus a few holes. But we are worried about extending over those holes, so we need to keep track of their geometry. Our neighbourhood of the 2-skeleton may be small, but still we can take a ball inside each 3-cell, containing the hole, and its curvatures (which may be very flat along the faces) will be bounded below. This is a "simple assertion":

**Lemma 4.1** *Let $\sigma$ be a 3-simplex of diameter $d$. For any $\lambda > 0$, there exists and embedded ball $B \subset \sigma$ such that its boundary is contianed in a $\lambda$-neighbourhood of $\delta\sigma$ and the normal curvatures of $\partial B$ are everywhere $\geq 8\lambda/(4\lambda^2 + d^2)$.*

But because we have such a fine subdivision, $\xi_t$ doesn't change very much. And since we can bound $||\xi_t'||$ in terms of $||\xi_t||$, $\xi_t'$ doesn't change very much either. So the characteristic foliation on the boundary $\partial B$ of our ball $B$ will turn out to be rather simple.

How simple? Well, let us digress for a minute and consider one-dimensional foliations $\mathcal{F}$ on $S^2$, in particular those with precisely two elliptic singular points, at the poles. (Our situation will have tow such poles.) Such a foliation is *simple* if all its limit cycles are isolated and placed on parallels between the foci. It is *almost horizontal* if there is a transversal to $\mathcal{F}$ connecting the poles. (When you draw a picture, an almost horizontal foliation "never turns around" between limit cycles.) Almost horizontal is very nice, because an almost horizontal foliation gives a holonomy map $h(\mathcal{F}) : I \to I$ which is a diffeomorphism of the interval. (Consider a transversal and a return map.) Almost horizontal is very nice also, because the present situation is almost horizontal!

Why? Another "simple assertion":

**Lemma 4.2** *Let $S \subset \mathbb{R}^3$ be an embedded 2-sphere with all normal curvatures $\geq K > 0$. Let $\xi$ be a contact structure near $S$ with $||\xi|| < K$. Then $\xi$ is almost horizontal near $S$.*

With a sufficiently fine subdivision, we have normal curvatures everywhere arbitrarily high while keeping our neighbourhood sufficiently small for the previous part of the proof to work(if you check the dependencies in the previous part of the proof, the neighbourhood width $\lambda$ was of the form $\epsilon d$, where $\epsilon$ depended only on $\delta/d$ and $\alpha$). So this lemma applies. As to why the lemma is true, well, being not almost horizontal (that is, having a "turn around" in the foliation) implies the contact structure turning quite fast; faster than the curvature, it seems.

It turns out that the topological type of the characteristic foliation on the sphere is all that matters for our purposes.

**Lemma 4.3** *Let $\xi$ be a simple contact structure near the boundary $S = \partial B$ of the 3-ball $B$. The extendability of $\xi$ as a contact structure to $B$ depends only on the topological type of the foliation $S_\xi$.*

Why is this true? It's (what later became) a standard perturbation argument. Take two contact structures $\xi_t$, $\xi'_t$ near $S$. We take $L \subset S$ to be a union of transversals and limit cycles of both characteristic foliations. [Taking transversals seems to imply we're talking about almost horizontal, rather than simple, contact structures near $S$. Hmm.] Let $N$ be a tubular neighbourhood of $L$ (in $\mathbb{R}^3$). We can get a contactomorphism $g$ on $S \backslash N \longrightarrow S \backslash N$, since these are just disks with standard foliations. We extend this diffeomorphism to $S$, remaining constant on $L$. Now $g$ will not necessarily be a contactomorphism on all of $S$; but the characteristic foliation will be $C^1$-close to what is required. We now perturb $C^0$-perturb $g$, to embed $S$ in $B$, and $C^1$-adjust the characteristic foliation.

So this now gives:

**Lemma 4.4** *Let $M, A, K, L, \xi_t$ be as above. There exist disjoint 3-balls $B_1, \ldots, B_N$ which avoid $A \cup \Delta$ and distributions $\xi'_t$ on $M$, defined for all $t$ in the parameter space $K$, such that*

*(i) $\xi'_t$ and $\xi_t$ agree on $(A \cup \Delta) \times K \cup M \times L$;*

*(ii) for all $t \in K$, $\xi'_t$ is contact everywhere except the interiors of the $B_i$;*

*(iii) for all $t \in K$, $\xi'_t$ is almost horizontal near every $\partial B_i$;*

*(iv) for all $t \in K$, $\xi'_t$ is $C^0$-close to $\xi_t$;*

Notice $A$ sneakily became $A \cup \Delta$ here, but that is no big deal. We just treat the overtwisted disk $\Delta$ as part of $A$.

# 5 Step III: Making one hole and filling it

We can connect sum simple foliations on spheres by cutting off neighbourhoods of poles and gluing and smoothing the glued foliations. We order our balls arbitrarily $B_1, \ldots, B_N$ and take for $B_0$ a small ball containing the overtwisted disk $\Delta$. Now we connect the north pole of each $B_i$ with the south pole of the next, by disjoint embedded curves $l_t^i$.

We need all of the $l_t^i$ to be transverse, else our connected balls will not have standard characteristic foliations. So we need to be able to perturb curves to be transverse; in fact, we need to be able to perturb families of curves to be families of transverse curves. It is not so difficult to do this for one curve: just take enough wavefronts with the right gradients at the right points. It's not so difficult to do this in families.

Having done that, we have connected the balls, so we have a contact structure with one hole. If we can extend the contact structure over the ball, and extend homotopies of plane fields to homotopies of contact structures over the ball, we are done. But this is not difficult. We know that the topological type of the almost horizontal foliation is all that matters. In fact, given an almost horizontal foliation, we can construct an explicit model of a solid of revolution in $\mathbb{R}^3$ which gives that almost horizontal foliation. (There's a picture in [1].) These can be homotoped, no problem, and remaining constant on $A$.

# 6 Corollaries and related results

The classification of overtwisted contact structures has an interesting corollary about extending contact embeddings of $B^3 \longrightarrow S^3$ to $S^3 \longrightarrow S^3$, Eliashberg's 1.6.2 in [1].

**Theorem 6.1** *There exists an overtwisted structure $\xi$ on $S^3$, a ball $B \subset S^3$ and a contact embedding $\psi : (B, \xi) \longrightarrow (S^3, \xi)$ such that $\psi$ cannot be extended to a contact diffeomorphism $S^3 \longrightarrow S^3$. Hence $\psi$ cannot be connected with the inclusion $B \longrightarrow S^3$ by a contact isotopy. (If it could, the isotopy would easily extend to an isotopy of diffeomorphisms of $S^3$ and one end of this would be an extension of $\psi$.)*

A very sketchy idea of the proof is to do Lutz twisting twice to get a contact structure that has been made overtwisted: but the Lutz twisting once (about $B$, giving $\zeta$) or twice (about $B$ and $B'$, giving $\zeta'$) remains in the same homotopy class, and hence $\zeta, \zeta'$ are isotopic, so also contactomorphic via $h : S^3 \longrightarrow S^3$. The two balls $B, B'$ containing the Lutz twistings are clearly contactomorphic. Mapping one to the other, and then applying the contactomorphism $h$ gives a contact embedding $\psi$. But then outside $B$ and its image, the contact structure is tight on one side, and not on the other.

# References


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[2] M. Gromov, Stable mappings of foliations into manifolds, Izv. Acad. Naur SSSR, Ser. mat. 33, 1206–1209 (1969).

[3] R. Lutz, Structures de contact sur les fibrés principaux en cercles de dimension 3. Ann. Inst. Fourier, 3, 1–15 (1977).